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Fast Galerkin methods for thermal regenerator modelling

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Abstract—The use of the Legendre orthogonal polynomials provides a significant improvement in the robustness and simplicity of the Galerkin series expansion methods for the solution of the classical Nusselt integral equations. These equations describe the steady state temperature behaviour of a thermal regenerator. A novel, economical method is offered for Galerkin techniques for the non-symmetric case.

INTRODUCTION

THE GALERKIN methods described in this paper, are used for the numerical solution of the integral equation

$$F(\Lambda-\xi)+e^{-\Pi}F(\xi)+\int_0^{\xi}K(\xi-\varepsilon)F(\varepsilon)\,d\varepsilon=1 \quad (1)$$

and the pair of simultaneous integral equations

$$F'(\xi')=e^{-\Pi'}F(\xi')+\int_{\xi'}^{\Lambda'}K'(\varepsilon-\xi')F(\varepsilon)\,d\varepsilon \quad (2)$$

$$(1-F(\xi))=e^{-\Pi}(1-F'(\xi))+\int_0^{\xi}K(\xi-\varepsilon)(1-F'(\varepsilon))\,d\varepsilon. \quad (3)$$

This approach was first suggested by Baclic [1] in 1985 for the solution of the symmetric regenerator problem represented here by equation (1). The *kernel* of these integral equations is defined by

$$K(x)=\frac{-iJ_1(2i(x\Pi)^{1/2})}{(x\Pi)^{1/2}}\Pi e^{-x-\Pi} \quad (4)$$

where $iJ_1(iy)$ is a real valued function with complex argument iy, where $i^2=-1$ and J_1 is the Bessel function of the first type and of first order. Iliffe [2] and Hausen [3] offered a similar set of integral equations. These had been devised originally by Nusselt [4, 5].

The equations describe the relationship between the distance variation of the packing temperature, $F(\xi)$, for $0\leqslant\xi\leqslant\Lambda$, at the end of the *hot period* of thermal regenerator operation and the corresponding variation $F'(\xi')$, for $0\leqslant\xi'\leqslant\Lambda'$, for the end of the *cold period.*

It is conventional to introduce dimensionless scales, (see Hausen [6]) and, in particular, the length of the regenerator for the hot period is denoted by Λ, in the cold period by Λ'. These parameters were called the *reduced length* by Hausen. The corresponding dimensionless time parameters are the *reduced period*, Π, for the hot period and Π' for the cold.

Equations (2) and (3) deal with the general non-symmetric case where $\Lambda\neq\Lambda'$ and/or $\Pi\neq\Pi'$. The Galerkin treatment of this case was first attempted by Balic and Dragutinovic [7] in 1991. Note that equation (2) assumes that gas flows through the packing in the cold period, entering the regenerator at $\xi'=\Lambda'$ and departing at position $\xi'=0$ whereas in the work of Iliffe [2] and Hausen [3] the equation is written with the gas entering at $\xi'=0$ and leaving at $\xi'=\Lambda'$.

Baclic's first paper [1] uses equation (1) which exploits the symmetry of the case where $\Lambda=\Lambda'$ and $\Pi=\Pi'$ for which it can be shown that

$$F'(\Lambda-\xi)+F(\xi)=1. \quad (5)$$

Baker [8] describes how such integral equations are known as Volterra equations of the second kind. The Galerkin techniques are *series expansion* methods, in which we seek to approximate $F(\xi)$ and $F'(\xi')$ by $\Psi(\xi)$ and $\Psi'(\xi)$, respectively, where

$$\Psi(\xi)=\sum_{j=0}^{n}\alpha_j\phi_j(\xi) \quad (6)$$

and

$$\Psi'(\xi')=\sum_{j=0}^{n}\beta_j\phi_j(\xi'). \quad (7)$$

The series expansion solution consists of vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ where $\boldsymbol{\alpha}=[\alpha_0,\alpha_1,\alpha_2,\ldots,\alpha_n]^T$ and $\boldsymbol{\beta}=[\beta_0,\beta_1,\beta_2,\ldots,\beta_n]^T$.

The series expansions (6) and (7) embody a set of *linearly independent* functions $\{\phi_j(\xi)\,|\,j=0,1,2,\ldots,n\}$. Without loss of generality, substitution of expansion (6) into equation (1) yields

NOMENCLATURE

A	square $(n+1)\times(n+1)$ matrix, equation (30)
F	dimensionless solid temperature on [0, +1] scale
$\mathbf{h}$	vector of order $(n+1)=[\Lambda, 0, 0, \ldots, 0]^T$
$K(\xi-\varepsilon)$	kernel of integral equations
$P_i(x)$	Legendre polynomial of degree i
$Q_i(x)$	modified Legendre polynomial of degree i, equation (19)
T	solid temperature [K]
t	gas temperature [K].

Greek symbols

$\boldsymbol{\alpha}$	vector of order $(N+1)$, see equation (6)
$\boldsymbol{\beta}$	vector of order $(N+1)$, see equation (7)
$\delta_{i,j}$	Kronecker delta = 1 if $i=j$, = 0 if $i\neq j$
ε	dummy variable
η_{REG}	thermal ratio
Λ	reduced length
ξ	dimensionless distance from the regenerator entrance where $0\leqslant\xi\leqslant\Lambda$
Π	reduced period.

Superscripts

prime	refers to the cold period
(p)	refers to the pth cycle.

Subscript

in	refers to the regenerator inlet.

All other symbols are defined explicitly within the text.

$$\sum_{j=0}^{n}\alpha_j\left\{\phi_j(\Lambda-\xi)+\mathrm{e}^{-\Pi}\phi_j(\xi)+\int_0^{\xi}K(\xi-\varepsilon)\,\phi_j(\varepsilon)\,\mathrm{d}\varepsilon\right\}=1. \quad (8)$$

The integral equation (1) can be written in more general terms as

$$\Omega(F(\xi))=0 \quad (9)$$

which would then adopt the form, for $0\leqslant\xi\leqslant\Lambda$,

$$\Omega(\Psi(\xi))=0 \quad (10)$$

after the substitution of expansion (6) into equation (1) provided $\Psi(\xi)$ were an exact solution to the integral equation. However, since $\Psi(\xi)$ is only an approximate solution, then criteria must be agreed as to how good a solution it might be. This manifests itself in the criteria used to ascertain the coefficients $\{\alpha_k\,|\,k=0,1,2,\ldots,n\}$. In the Galerkin method, we seek to *minimise* $\Omega(\Psi(\xi))$ on the interval $0\leqslant\xi\leqslant\Lambda$. In so doing, the vector $\boldsymbol{\alpha}$ is determined by a process not unlike the least squares technique, by solving the equations

$$\int_0^{\Lambda}\Omega(\Psi(\xi))\,\phi_j(\xi)\,\mathrm{d}\xi=0 \quad (11)$$

for $j=0, 1, 2, \ldots, n$. This reduces to a form:

$$A\,\boldsymbol{\alpha}=\mathbf{h}. \quad (12)$$

A is a square $(n+1)\times(n+1)$ matrix and $\mathbf{h}$ is a vector of order $n+1$. These will be defined later.

This is in contrast to the collocation method which has parallels with interpolation. Willmott and Knight [9] describe how the vector $\boldsymbol{\alpha}$ is determined by requiring

$$\Omega(\Psi(\xi_k))=0 \quad (13)$$

at $n+1$ distinct collocation points, $\{\xi_k\,|\,k=0, 1, 2, \ldots, n\}$ all on the interval $0\leqslant\xi_k\leqslant\Lambda$,

In this paper, we propose another set of functions $\{\phi_j(\xi)\,|\,j=0, 1, \ldots, n\}$ to those proposed previously by Baclic [1] and, in particular, introduce the idea of using functions with two useful properties, namely their values lie in a range which contains closely that occupied by $F(\xi)$ and $F'(\xi')$, and, equally important, they are *orthogonal*. This leads to a simplification of the methodology. In addition, we exploit a method for handling the general case represented by equations (2) and (3), previously set out by Willmott and Knight [9]. In so doing, significant economies are introduced over the methodology set out by Baclic and Dragutinovic [7] for the solution of the *non-symmetric* problem.

THERMAL REGENERATOR OPERATION AND ITS MODELLING

Heat is transferred in a regenerator from a hot gas, typically the combustion products from a fuel gas used to fire a furnace or a boiler, to a cold gas, often the air required for the combustion of that fuel. This is realised by passing the hot gas through the channels of a packing in which the thermal energy is then stored. This process continues for the duration of the *hot period* after which the cold gas passes through the same channels of the packing in the contraflow direction during the *cold period* when the heat is recovered from the packing and the cold gas is heated up. In this work, we treat a *cycle of operation* as consisting of a cold period followed by a hot period. After many cycles, a regenerator realises *cyclic* or *dynamic equilibrium*.

Equation (2) relates the temperature distribution $F(\xi')$ at the start of the *cold period* at cyclic equi-

librium to that at the end of the cold period, namely $F'(\xi')$. Similarly, equation (3) relates the temperature distribution at the beginning of the hot period of an equilibrium cycle to that of the finish of the period.

This is the model which is used in this work. The assumptions embodied there are described in detail by Schmidt and Willmott [10] and also by Hausen [11] in his classical treatise on this subject. In essence, this is a so-called linear model in which it is assumed that the thermophysical properties of the gas and solid are temperature independent, in which the inlet gas temperatures and the gas flow rates in each period of regenerator operation do not vary with time. The solid temperatures $T(\xi)$ vary, at cyclic equilibrium, in the range

$$t'_{in} < T(\xi) < t_{in}$$

where t'_{in} and t_{in} are the cold and hot inlet gas temperatures to the regenerator. We generate the $[0, +1]$ dimensionless temperature scales using the transformation:

$$F(\xi) = \frac{T(\xi) - t'_{in}}{t_{in} - t'_{in}}.$$

This range for $F(\xi)$ is exploited in our choice of the Legendre polynomials as linearly independent functions in the series expansions (6) and (7).

ORTHOGONAL POLYNOMIALS

In the approach described in this paper, a set of *linearly independent* functions $\{\rho_j(x) \mid j = 0, 1, 2, \ldots, n\}$ are selected which possess the property of *orthogonality*, that is

$$\int_{-1}^{+1} \rho_i(x)\,\rho_j(x)\,dx = 0 \text{ for } i \neq j \tag{14}$$

and

$$\int_{-1}^{+1} \rho_j(x)^2\,dx \neq 0. \tag{15}$$

These functions $\{\rho_j(x) \mid j = 0, 1, 2, \ldots, n\}$ are polynomials of degree j of the independent variable x where $-1 \leqslant x \leqslant +1$. In addition, they possess the useful property that they can be defined in terms of a recurrence relationship of the form

$$\rho_{j+1}(x) = (a_j x + b_j)\rho_j(x) + c_{j-1}\rho_{j-1}(x) \tag{16}$$

where $\rho_0(x)$ is a constant and $\rho_1(x)$ is a linear function of x. The particular functions we choose are the *Legendre polynomials* $P_j(x)$ which may be defined by the recurrence relationship, which can be readily used within a piece of software,

$$P_{j+1}(x) = \frac{2j+1}{j+1} x P_j(x) - \frac{j}{j+1} P_{j-1}(x) \tag{17}$$

with $P_0(x) = 1$ and $P_1(x) = x$.

It can be shown that

$$\int_{-1}^{+1} P_j(x)^2\,dx = \frac{2}{2j+1} \neq 0. \tag{18}$$

The property

$$-1 \leqslant P_j(x) \leqslant +1$$

means that the range for $P_j(x)$ contains closely the range for $F(\xi)$ and $F'(\xi)$, so that, combined with the recurrence relation (17) and the orthogonality property, the polynomials $\{P_j(x) \mid j = 0, 1, 2, \ldots, n\}$ become ideal candidates as possible linear functions for the expansions (6) and (7).

The Legendre polynomials are defined in terms of the variable x on the interval $[-1, +1]$ whereas the regenerator model operates with the variable ξ on the interval $[0, \Lambda]$. This is overcome by use of the change of variable

$$x = \frac{2\xi - \Lambda}{\Lambda}.$$

It will be seen that, when $\xi = 0$, then $x = -1$, whereas, when $\xi = \Lambda$, then $x = +1$.

In order to simplify the notation, we introduce the polynomials $\{Q_j(\xi) \mid j = 0, 1, 2, \ldots, n\}$ where

$$Q_j(\xi) = P_j\left(\frac{2\xi - \Lambda}{\Lambda}\right) \tag{19}$$

so that the orthogonality relation (14), for example, takes the form

$$\frac{2}{\Lambda}\int_0^{\Lambda} Q_i(\xi)\,Q_j(\xi)\,d\xi = 0 \text{ for } i \neq j \tag{20}$$

and the series solution to equation (1) takes the form

$$\Psi(\xi) = \sum_{j=0}^{n} \alpha_j Q_j(\xi). \tag{21}$$

There is one additional property of these orthogonal polynomials which proves to be extremely useful in simplifying the method of solution of the regenerator problem. The orthogonality relation (20) is applicable for the case where $i \neq j$ and where $j = 0$. Since

$$P_0(x) = Q_0(\xi) = 1$$

it follows that

$$\frac{2}{\Lambda}\int_0^{\Lambda} Q_i(\xi)\,d\xi = 0 \text{ for } i \geqslant 1. \tag{22}$$

An important parameter describing the overall performance of the regenerator is the *thermal ratio*, η_{REG}. This is defined by

$$\eta_{REG} = \frac{1}{\Pi}\int_0^{\Lambda} (F(\xi) + F(\Lambda - \xi) - 1)\,d\xi \tag{23}$$

for the symmetric case. Using equation (5), this reduces to:

$$\eta_{\mathrm{REG}} = \frac{1}{\Pi}\int_0^{\Lambda} (2F(\xi)-1)\,\mathrm{d}\xi. \tag{24}$$

If we now replace $F(\xi)$ by $\Psi(\xi)$, we obtain

$$\eta_{\mathrm{REG}} = \frac{1}{\Pi}\int_0^{\Lambda} (2\sum_{j=0}^{n} \alpha_j Q_j(\xi)-1)\,\mathrm{d}\xi \tag{25}$$

where the last n terms within the summation disappear upon integration by virtue of the orthogonality property. In other words, equation (25) is greatly simplified and we are left with

$$\eta_{\mathrm{REG}} = \frac{1}{\Pi}\int_0^{\Lambda} (2\alpha_0 Q_0(\xi)-1)\,\mathrm{d}\xi = \frac{1}{\Pi}\int_0^{\Lambda} (2\alpha_0-1)\,\mathrm{d}\xi. \tag{26}$$

It follows that the thermal ratio can be computed, knowing only the value of α_0 and

$$\eta_{\mathrm{REG}} = \frac{\Lambda}{\Pi}(2\alpha_0-1). \tag{27}$$

We shall show later that an equally simple form of the thermal ratio can be found for the unsymmetric case. It is important to note that equation (27) can be applied both in the case of where α_0 is found by the method of Galerkin and in that where it is determined by the method of collocation.

APPLICATION OF THE GALERKIN METHOD FOR THE SYMMETRIC CASE

We have indicated previously that the coefficients $\{\alpha_k \mid k = 0, 1, 2, \ldots, n\}$ are determined by solving the equations

$$\int_0^{\Lambda} \Omega(\Psi(\xi))\,\phi_j(\xi)\,\mathrm{d}\xi = 0 \tag{11}$$

for $j = 0, 1, 2, \ldots, n$. Baclic [1] explains that, in so doing, we determine the values of $\{\alpha_k \mid k = 0, 1, 2, \ldots, n\}$ in such a way that $\Omega(\Psi(\xi))$ is minimised on the interval $0 \leqslant \xi \leqslant \Lambda$. We now expand equation (11) and obtain

$$\int_0^{\Lambda} \left(\sum_{j=0}^{n} \alpha_j \left\{ Q_j(\Lambda-\xi) + \mathrm{e}^{-\Pi} Q_j(\xi) + \int_0^{\xi} K(\xi-\varepsilon)\,Q_j(\varepsilon)\,\mathrm{d}\varepsilon \right\} - 1 \right) Q_i(\xi)\,\mathrm{d}\xi = 0 \tag{28}$$

for $i = 0, 1, 2, \ldots, n$. The order of the integration and the summation can be interchanged so that equation (28) is modified to become

$$\sum_{j=0}^{n} \alpha_j \left\{ \int_0^{\Lambda} \{ Q_j(\Lambda-\xi) + \mathrm{e}^{-\Pi} Q_j(\xi) + \int_0^{\xi} K(\xi-\varepsilon)\,Q_j(\varepsilon)\,\mathrm{d}\varepsilon \}\, Q_i(\xi)\,\mathrm{d}\xi \right\} = \int_0^{\Lambda} Q_i(\xi)\,\mathrm{d}\xi \tag{29}$$

for $i = 0, 1, 2, \ldots, n$. The vector $\boldsymbol{\alpha}$ is now determined by solving the set of linear equations

$$A\boldsymbol{\alpha} = \mathbf{h} \tag{12}$$

where A is a square $(n+1) \times (n+1)$ matrix and $\mathbf{h}$ is a vector of order $n+1$. It will be seen that the matrix A is given by

$$A = \left[\int_0^{\Lambda} \left\{ Q_j(\Lambda-\xi) + \mathrm{e}^{-\Pi} Q_j(\xi) + \int_0^{\xi} K(\xi-\varepsilon)\,Q_j(\varepsilon)\,\mathrm{d}\varepsilon \right\} Q_i(\xi)\,\mathrm{d}\xi \right] \tag{30}$$

for $i, j = 0, 1, 2, \ldots, n$, while the vector $\mathbf{h}$ takes the form

$$\mathbf{h} = \left[\int_0^{\Lambda} Q_i(\xi)\,\mathrm{d}\xi \right] \tag{31}$$

for $i = 0, 1, 2, \ldots, n$. It can be shown for the Legendre polynomials that

$$Q_j(\Lambda-\xi) = (-1)^j Q_j(\xi).$$

Further, the property given by equation (18) becomes

$$\int_0^{\Lambda} Q_j(\xi)^2\,\mathrm{d}\xi = \frac{\Lambda}{2}\frac{2}{2j+1} \neq 0. \tag{32}$$

These and the orthogonality properties of the polynomials $\{Q_i(\xi)\}$ lead to a good deal of simplification of the elements of the matrix A. Indeed, the first two terms in the off-diagonal elements (where $i \neq j$) of the matrix, as shown in equation (30), are annihilated as a consequence of the $\{Q_i(\xi)\}$ being *orthogonal*. The calculation of the diagonal elements of A is facilitated by equation (32). It follows that

$$A_{i,j} = \int_0^{\Lambda} \left\{ \int_0^{\xi} K(\xi-\varepsilon)\,Q_j(\varepsilon)\,\mathrm{d}\varepsilon \right\} Q_i(\xi)\,\mathrm{d}\xi \text{ for } i \neq j \tag{33}$$

and for the diagonal elements of A, we have

$$A_{i,i} = \frac{\Lambda}{2}\frac{2}{2i+1}\{(-1)^{-i} + \mathrm{e}^{-\Pi}\} + \int_0^{\Lambda} \left\{ \int_0^{\xi} K(\xi-\varepsilon)\,Q_i(\varepsilon)\,\mathrm{d}\varepsilon \right\} Q_i(\xi)\,\mathrm{d}\xi. \tag{34}$$

The elements of the vector $\mathbf{h}$ are similarly modified and thereby greatly simplified. In particular,

$$h_i = 0 \text{ for } i \geqslant 1 \tag{35}$$

following equation (22). The top element of $\mathbf{h}$ is simply

$$h_0 = \int_0^{\Lambda} Q_0(\xi)\,\mathrm{d}\xi = \int_0^{\Lambda} \mathrm{d}\xi = \Lambda. \tag{36}$$

SOME COMPUTATIONAL CONSIDERATIONS

Although the introduction of the *orthogonal* polynomials enables the form of the matrix A, the vector $\mathbf{h}$ and the thermal ratio η_{REG} to be greatly simplified, it still remains the case that it is necessary to evaluate the integrals in equations (33) and (34), namely

$$\int_0^{\Lambda}\left\{\int_0^{\xi} K(\xi-\varepsilon)\,Q_j(\varepsilon)\,\mathrm{d}\varepsilon\right\}Q_i(\xi)\,\mathrm{d}\xi$$

for all the elements of the matrix A. Baclic [1] has suggested that the integral

$$\int_0^{\xi} K(\xi-\varepsilon)\varepsilon^j\,\mathrm{d}\varepsilon$$

can be computed easily by regarding the integral as a convolution of the functions $K(\xi-\varepsilon)$ and ε^j. He proposes a recurrence formula whereby this integral can be evaluated. We have examined this strategy and have found that is economical provided the ratio of the reduced length to the reduced period, Λ/Π is large (>10) and $\Lambda<100$. In other cases, the convergence of Baclic's recurrence formula is very slow. In an extreme case where $\Lambda=1000$ and $\Lambda/\Pi=0.01$, almost 2000 terms in the recurrence were required to realise convergence. Certainly 100 terms are required, at least, for $\Lambda=100$ for $\Lambda/\Pi\leqslant 10$.

This strategy, nevertheless, can be applied to the case where it is required to evaluate

$$\int_0^{\xi} K(\xi-\varepsilon)\,Q_j(\varepsilon)\,\mathrm{d}\varepsilon$$

simply by decomposing the Legendre polynomial into its component parts and integrating term by term. For example,

$$P_2(x)=\frac{1}{2}(3x^2-1),\ P_3(x)=\frac{1}{2}(5x^3-3x)\ \text{and}$$

$$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$$

can be converted into polynomials in ξ using the change of variable

$$x=\frac{2\xi-\Lambda}{\Lambda}.$$

In the Appendix, Arthurs and Willmott offer another approach. Alternatively, the integrals can be evaluated by numerical quadrature.

We have implemented the Galerkin method for the symmetric case using the modified Legendre polynomials $\{Q_j(\xi)\,|\,j=0,1,2,\ldots..\}$ and the trial functions $\{\xi^j/j!\,|\,j=0,1,2,\ldots..\}$ suggested by Baclic. Our results suggest that the number of terms required in $\Psi(\xi)$ for five figure accuracy in the thermal ratio, η_{REG}, using the $Q_j(\xi)$ polynomials is *never* greater than the number needed when using the Baclic trial functions. Indeed, for $\Lambda>100$ and $\Lambda/\Pi<0.1$, the use of the Legendre polynomials is more economical. For example, for $\Lambda=1000$ and $\Pi/\Lambda=0.05$, a polynomial of degree 10 is required when using $\{\xi^j/j!\,|\,j=0,1,2,\ldots..\}$ whereas a polynomial of degree 7 is needed if the Legendre polynomials are used.

THE NON-SYMMETRIC CASE

A computational problem arises in the non-symmetric case. Solution of equation (12) involves the solution of $n+1$ simultaneous equations. The approach of Baclic [7] to equations (2) and (3) involve the solution of $2n+2$ equations, where the solution vector is a concatenation of the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, defined by equations (6) and (7). This is equivalent to the approach by Nahavandi and Weinstein [12] to the collocation method. This doubling of the number of equations results in the computational effort being multiplied by approximately eight. Willmott and Knight [9] have indicated how this problem might be overcome for both the Galerkin and collocation approaches to the integral equations (2) and (3). The matrix method which they described is outlined below.

THE MATRIX METHOD

Equation (2) can be re-written for the cold period in the pth cycle:

$$F'^{(p)}(\xi')=e^{-\Pi'}F^{(p)}(\xi')+\int_{\xi'}^{\Lambda'}K'(\varepsilon-\xi')\,F^{(p)}(\varepsilon)\,\mathrm{d}\varepsilon. \tag{37}$$

Inserting expansions (6) and (7) and denoting the vectors for the pth cycle by $\boldsymbol{\alpha}^{(p)}$ and $\boldsymbol{\beta}^{(p)}$ yields

$$\sum_{j=0}^{n}\beta_j^{(p)}\phi_j(\xi')=\sum_{j=0}^{n}\alpha_j^{(p)}\left\{e^{-\Pi'}\phi_j(\xi')+\int_{\xi'}^{\Lambda'}K'(\varepsilon-\xi')\,\phi_j(\varepsilon)\,\mathrm{d}\varepsilon\right\}. \tag{38}$$

Application of Galerkin's method to equation (38) yields a matrix equation, namely

$$\Phi'\boldsymbol{\beta}^{(p)}=\Gamma'\boldsymbol{\alpha}^{(p)} \tag{39}$$

where Φ' and Γ' are $(0..n)\times(0..n)$ matrices. These are defined by

$$\Phi'=\left[\int_0^{\Lambda'}\phi_j(\xi')\,\phi_i(\xi')\,\mathrm{d}\xi'\right] \tag{40}$$

$$\Gamma'=\left[\int_0^{\Lambda'}\left\{e^{-\Pi'}\phi_j(\xi')+\int_{\xi'}^{\Lambda'}K'(\varepsilon-\xi')\times\phi_j(\varepsilon)\,\mathrm{d}\varepsilon\right\}\phi_i(\xi')\,\mathrm{d}\xi'\right]. \tag{41}$$

Upon setting $\phi_i(\xi')=Q_i(\xi')$, the matrix Φ' is greatly simplified and becomes diagonal in form as a consequence of the *orthogonality* of the functions

$\{Q^i(\xi') \mid i = 0, 1, 2, \ldots\}$. The diagonal elements take the form

$$\Phi'_{i,i} = \frac{\Lambda'}{2}\frac{2}{2i+1} \quad \text{for } i = 0, 1, 2, \ldots, n. \tag{42}$$

The matrix Γ' is simplified in a similar manner. The off-diagonal elements assume the form

$$\Gamma'_{i,j} = \int_0^{\Lambda'} \left[\int_{\xi'}^{\Lambda'} K'(\varepsilon - \xi') Q_j(\varepsilon)\, d\varepsilon \right] Q_i(\xi')\, d\xi' \quad \text{for } i \neq j \tag{43}$$

while the diagonal elements are

$$\Gamma'_{i,i} = \frac{e^{-\Pi'}\Lambda'}{2i+1} + \int_0^{\Lambda'} \left\{ \int_{\xi'}^{\Lambda'} K'(\varepsilon - \xi') \times Q_i(\varepsilon)\, d\varepsilon \right\} Q_i(\xi')\, d\xi' \quad \text{for } i = 0, 1, 2, \ldots, n. \tag{44}$$

The vector $\boldsymbol{\beta}^{(p)}$ can be computed from $\boldsymbol{\alpha}^{(p)}$ using

$$\boldsymbol{\beta}^{(p)} = \Phi'^{-1}\Gamma'\boldsymbol{\alpha}^{(p)}. \tag{45}$$

This involves only matrix multiplications since Φ' is diagonal, see equation (42).

Equation (3) for the hot period in the pth cycle is developed in a similar manner. It is somewhat re-arranged to yield

$$F^{(p+1)}(\xi) = e^{-\Pi} F'^{(p)}(\xi) + \int_0^{\xi} K(\xi - \varepsilon) F'^{(p)}(\varepsilon)\, d\varepsilon + \left\{ 1 - e^{-\Pi} - \int_0^{\xi} K(\xi - \varepsilon)\, d\varepsilon \right\}. \tag{46}$$

Substituting again expansions (6) and (7), this time into equation (46) generates

$$\sum_{j=0}^{n} \alpha_j^{(p+1)} \phi_j(\xi) = \sum_{j=0}^{n} \beta_j^{(p)} \left\{ e^{-\Pi} \phi_j(\xi) + \int_0^{\xi} K(\xi - \varepsilon) \phi_j(\varepsilon)\, d\varepsilon \right\} + \left\{ 1 - e^{-\Pi} - \int_0^{\xi} K(\xi - \varepsilon)\, d\varepsilon \right\}. \tag{47}$$

Application of Galerkin transformation to equation (47) produces an equation in matrix form which is

$$\Phi \boldsymbol{\alpha}^{(p+1)} = \Gamma \boldsymbol{\beta}^{(p)} + \zeta \tag{48}$$

$$\Phi = \left[\int_0^{\Lambda} \phi_j(\xi) \phi_i(\xi)\, d\xi \right] \tag{49}$$

$$\Gamma = \left[\int_0^{\Lambda} \left\{ e^{-\Pi} \phi_j(\xi) + \int_0^{\xi} K(\xi - \varepsilon) \phi_j(\varepsilon)\, d\varepsilon \right\} \phi_i(\xi)\, d\xi \right]. \tag{50}$$

The vector $\zeta = [\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_n]^T$ is defined by

$$\zeta_i = \int_0^{\Lambda} \left\{ 1 - e^{-\Pi} - \int_0^{\xi} K(\xi - \varepsilon)\, d\varepsilon \right\} \phi_i(\xi)\, d\xi. \tag{51}$$

Again, setting $\phi_i(\xi) = Q_i(\xi)$, the matrix Φ becomes diagonal in form with

$$\Phi_{i,i} = \frac{\Lambda}{2i+1} \text{ for } i = 0, 1, 2, \ldots, n. \tag{52}$$

The off-diagonal elements of Γ take the form

$$\Gamma_{i,j} = \int_0^{\Lambda} \left\{ \int_0^{\xi} K(\xi - \varepsilon) Q_j(\varepsilon)\, d\varepsilon \right\} Q_i(\xi)\, d\xi \tag{53}$$

while the diagonal elements are:

$$\Gamma_{i,i} = \frac{e^{-\Pi}\Lambda}{2i+1} + \int_0^{\Lambda} \left\{ \int_0^{\xi} K(\xi - \varepsilon) Q_i(\varepsilon)\, d\varepsilon \right\} Q_i(\xi)\, d\xi. \tag{54}$$

The elements of the vector ζ take the modified form:

$$\zeta_i = -\int_0^{\Lambda} \left\{ \int_0^{\xi} K(\xi - \varepsilon)\, d\varepsilon \right\} Q_i(\xi)\, d\xi \text{ for } i > 0 \tag{55a}$$

$$\zeta_0 = \Lambda(1 - e^{-\Pi}) - \int_0^{\Lambda} \int_0^{\xi} K(\xi - \varepsilon)\, d\varepsilon\, d\xi. \tag{55b}$$

Equation (48) can now be re-arranged to give

$$\boldsymbol{\alpha}^{(p+1)} = \Phi^{-1}\Gamma\boldsymbol{\beta}^{(p)} + \Phi^{-1}\zeta. \tag{56}$$

Equation (57) is obtained by the substitution for $\boldsymbol{\beta}^{(p)}$ from equation (45); it takes the form

$$\boldsymbol{\alpha}^{(p+1)} = \Phi^{-1}\Gamma\Phi'^{-1}\Gamma'\boldsymbol{\alpha}^{(p)} + \Phi^{-1}\zeta. \tag{57}$$

If $\Phi^{-1}\Gamma\Phi'^{-1}\Gamma'$ is denoted by M and $\Phi^{-1}\zeta$ by $\boldsymbol{v}$, then equation (57) becomes

$$\boldsymbol{\alpha}^{(p+1)} = M\boldsymbol{\alpha}^{(p)} + \boldsymbol{v}. \tag{58}$$

At cyclic equilibrium, $\boldsymbol{\alpha}^{(p)} = \boldsymbol{\alpha}^{(p+1)} = \boldsymbol{\alpha}$ in which case, equation (58) takes the form

$$(I - M)\boldsymbol{\alpha} = \boldsymbol{v}$$

and the vector $\boldsymbol{\alpha}$ located by solving the set of $n+1$ linear equations, namely

$$\boldsymbol{\alpha} = (I - M)^{-1}\boldsymbol{v}. \tag{59}$$

Once $\boldsymbol{\alpha}$ has been found, the vector $\boldsymbol{\beta}$ is obtained from

$$\boldsymbol{\beta} = \Phi'^{-1}\Gamma'\boldsymbol{\alpha}. \tag{60}$$

The elegance of this approach lies in the fact that the matrices Φ and Φ' are diagonal and that therefore the matrix M and the vector $\boldsymbol{v}$ can be formed using matrix multiplications alone. Pre-multiplication by a diagonal matrix such as Φ^{-1} is a trivial operation. It is only in the calculation of the vector $\boldsymbol{\alpha}$ using equation (59) that it is necessary to solve a set of simultaneous linear equations. Again, the computation of the vector $\boldsymbol{\beta}$ employing equation (60) involves only matrix multiplications.

These properties, which are a consequence of the orthogonality of the functions $\{Q_i(\xi) | i = 0, 1, 2, \ldots\}$,

added to the fact that equation (59) involves the solution of only $n+1$ instead of $2n+2$ equations, makes this development of the Galerkin method particularly economical.

This Galerkin procedure described above represents a development of the matrix approach to regenerator simulations suggested recently by Willmott *et al.* [13] applied initially to the Iliffe method for solving equations (2) and (3) and subsequently by Willmott and Knight [9] to the method of collocation for the series solution of these same equations. This approach extends the work of Baclic and Dragutimoric [7] which encompasses the non-symmetric problem. The method embodied in this development represents an application of the Fast Galerkin Algorithm which is described by Delves and Mohamed [14].

THERMAL RATIO

The hot period thermal ratio is defined by the equation

$$\eta_{\mathrm{REG}} = \frac{\Lambda}{\Pi}\left[\frac{1}{\Lambda}\int_0^{\Lambda} F(\xi)\,\mathrm{d}\xi - \frac{1}{\Lambda'}\int_0^{\Lambda'} F'(\xi)\,\mathrm{d}\xi\right] \quad (61a)$$

while the cold period thermal ratio is given by

$$\eta'_{\mathrm{REG}} = \frac{\Lambda'}{\Pi'}\left[\frac{1}{\Lambda}\int_0^{\Lambda} F(\xi)\,\mathrm{d}\xi - \frac{1}{\Lambda'}\int_0^{\Lambda'} F'(\xi)\,\mathrm{d}\xi\right] \quad (61b).$$

The cold period η'_{REG} is modified upon substitution of the series solution of the integral equations, namely:

$$\eta'_{\mathrm{REG}} = \frac{\Lambda'}{\Pi'}\left[\frac{1}{\Lambda}\int_0^{\Lambda}\sum_{i=0}^{n}\alpha_i Q_i(\xi)\,\mathrm{d}\xi - \frac{1}{\Lambda'}\int_0^{\Lambda'}\sum_{i=0}^{n}\beta_i Q_i(\xi')\,\mathrm{d}\xi'\right]. \quad (62)$$

As in equation (25), the last n terms within each summation are annihilated as a consequence of the *orthogonality* of the functions $\{Q_i(\xi)|i=0,1,2,\ldots,n\}$. Equation (62) is immediately modified to take the form

$$\eta'_{\mathrm{REG}} = \frac{\Lambda'}{\Pi'}\left[\frac{1}{\Lambda}\int_0^{\Lambda}\alpha_0 Q_0(\xi)\,\mathrm{d}\xi - \frac{1}{\Lambda'}\int_0^{\Lambda'}\beta_0 Q_0(\xi')\,\mathrm{d}\xi'\right] \quad (63)$$

and

$$\eta'_{\mathrm{REG}} = \frac{\Lambda'}{\Pi'}\left[\frac{1}{\Lambda}\int_0^{\Lambda}\alpha_0\,\mathrm{d}\xi - \frac{1}{\Lambda'}\int_0^{\Lambda'}\beta_0\,\mathrm{d}\xi\right]. \quad (64)$$

The hot period η_{REG} is developed in exactly the same way. It follows that the thermal ratios can be calculated easily using

$$\eta_{\mathrm{REG}} = \frac{\Lambda}{\Pi}(\alpha_0-\beta_0), \quad \eta'_{\mathrm{REG}} = \frac{\Lambda'}{\Pi'}(\alpha_0-\beta_0) \quad (65)$$

and this corresponds to equation (27) which applies to the symmetric case.

CONCLUDING REMARKS

The use of the Legendre orthogonal polynomials greatly simplifies the Galerkin series solution of the integral equations (1)–(3). This in turn leads to economy and robustness in the method.

This becomes especially apparent in the matrix treatment of the *non-symmetric* case where solutions can be obtained by solving just one set of $n+1$ simultaneous equations (59), whereas earlier treatments have involved the solution of $2n+2$ equations. Some of the matrices are reduced to diagonal form as a consequence of the orthogonality property of the polynomials, thereby rendering the calculations even more economical.

Robustness is realised since the $[-1, +1]$ range of the polynomials $\{Q_i(\xi)\}$ closely contains the $[0, +1]$ range of the dimensionless temperature distributions $F(\xi)$ and $F'(\xi')$. In this way, the generation of large matrix elements is avoided, a problem only partially overcome in earlier work in this area.

It seems possible that this work might be extended to the use of other orthogonal polynomials although our early numerical experiments appear to suggest that any promise seemingly held out by the Chebyshev polynomials, for example, may not be realised. Further work in this area might be worthwhile although, at this juncture, it appears that the Legendre polynomials offer the greatest economy, speed and accuracy.

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APPENDIX

A. M. ARTHURS† and A. J. WILLMOTT

CALCULATION OF THE DOUBLE INTEGRAL IN EQUATIONS (33) AND (34)

The integral, which we denote by $C_{i,j}$ ($= A_{i,j}$ for $i \neq j$),

$$C_{i,j} = \int_0^{\Lambda} \left\{ \int_0^{\xi} K(\xi-\varepsilon)\, Q_j(\varepsilon)\, \mathrm{d}\varepsilon \right\} Q_i(\xi)\, \mathrm{d}\xi \qquad \text{(A1)}$$

can be simplified if we try to decouple the kernel $K(\xi-\varepsilon)$ from the orthogonal functions $Q_i(\xi)$. In essence, we create a new inner integral and then a new outer integral, both of which can be found analytically. Since the convolution integral is symmetric in arguments, equation (A1) can be rewritten as

$$C_{i,j} = \int_0^{\Lambda} \left\{ \int_0^{\xi} K(\varepsilon)\, Q_j(\xi-\varepsilon)\, \mathrm{d}\varepsilon \right\} Q_i(\xi)\, \mathrm{d}\xi \qquad \text{(A2)}$$

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from which it follows, if we *change the order of the integration*, that

$$C_{i,j} = \int_0^{\Lambda} \left\{ \int_{\varepsilon}^{\Lambda} Q_i(\xi)\, Q_j(\xi-\varepsilon)\, \mathrm{d}\xi \right\} K(\varepsilon)\, \mathrm{d}\varepsilon. \qquad \text{(A3)}$$

In equation (A3), the inner integral involves only a product of two Legendre polynomials. We can denote this integral as follows:

$$I_{i,j}(\varepsilon) = \int_{\varepsilon}^{\Lambda} Q_i(\xi)\, Q_j(\xi-\varepsilon)\, \mathrm{d}\xi \qquad \text{(A4)}$$

for $0 \leqslant \varepsilon \leqslant \Lambda$. There are special cases, namely when $\varepsilon = 0$ and when $\varepsilon = \Lambda$. In the first, the integral (A4) becomes

$$I_{i,j}(0) = \frac{\Lambda}{2} \frac{2}{2j+1} \delta_{i,j}$$

where $\delta_{i,j}$ is the Kronecker delta equal to 1 when $i = j$ and equal to zero for $i \neq j$. Clearly, also, the integral is zero in the second case where $\varepsilon = \Lambda$. More generally, the integral $I_{i,j}(\varepsilon)$ is a polynomial in ε of degree $i+j+1$ and can be evaluated directly.

The integral (A4) can thus be written

$$I_{i,j}(\varepsilon) = \sum_{k=0}^{i+j+1} a_k Q_k(\varepsilon). \qquad \text{(A5)}$$

We retain the use of the polynomials $Q_k(\varepsilon)$ to take advantage of the $[-1, +1]$ range occupied by them. The integral (A3) then becomes

$$C_{i,j} = \int_0^{\Lambda} \left\{ \sum_{k=0}^{i+j+1} a_k\, Q_k(\varepsilon) \right\} K(\varepsilon)\, \mathrm{d}\varepsilon. \qquad \text{(A6)}$$

This integral might be computed numerically, or, alternatively, it is possible to express $K(\varepsilon)$ as a power series in $(\varepsilon\Pi)$ multiplied by $\mathrm{e}^{-\varepsilon}$. This leaves the integral (A6) as an infinite series

$$C_{i,j} = \sum_{k=0}^{\infty} b_k \int_0^{\Lambda} Q_k(\varepsilon)\, \mathrm{e}^{-\varepsilon}\, \mathrm{d}\varepsilon. \qquad \text{(A7)}$$

The integral in (A7) can be found analytically and we are left with an infinite series for $A_{i,j}$ which, hopefully, is rapidly convergent. Examination of this expression will be the subject of future work.